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Stochastic evolution of a harmonic oscillator in a photon reservoir

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Abstract. The evolution of a harmonic oscillator in a photon reservoir is described by a quantum stochastic differential equation via the Wigner-Weisskopf approximation. The rotating wave approximation is shown to be equivalent to the requirement of quantum detailed balance and the irreversible evolution of the initial equilibrium state is explicitly computed.

1. Introduction

We consider the interaction of a single harmonic oscillator with an environment composed of several independent oscillators in thermal equilibrium. The interaction Hamiltonian contains both energy interchange and virtual-particle production terms.

The Wigner-Weisskopf approximation has the effect of replacing the environment by a finite-temperature free boson field. The dynamics of the interaction is then described by a unitary operator valued stochastic process which obeys a quantum stochastic differential equation [6]. This can be regarded as an intrinsically quantised generalisation of a stochastic differential equation of Ito type with respect to Brownian motion. In our case, the role of Brownian motion is played by a process composed of annihilation and creation operators which describe (respectively) energy losses and gains by the environment.

Just as in the classical case, where coarse graining of the process yields an irreversible evolution for the averages generated by a semi-elliptic partial differential operator, so in our case, when we average out the effects of the environment, we obtain a quantum dynamical semigroup describing the reduced evolution of the oscillator.

Our main result is the derivation of a formula describing the reduced evolution of the state of the oscillator under arbitrary 'weights' associated with energy interchange and virtual-particle production terms in the interaction Hamiltonian. We find that if both these terms appear with equal weight, then the semigroup drives the oscillator towards a state of 'infinite temperature'. As in the case of the two-level atom [3], the initial state of thermal equilibrium is preserved by the reduced dynamics if and only if all virtual-particle production terms are absent in the interaction Hamiltonian (i.e. the 'rotating wave approximation' [1] is imposed).

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2. Description of the oscillator and its environment

We consider an assembly of $(n+1)$ independent harmonic oscillators described by the Hamiltonian

$$H = \sum_{j=0}^{n} w_j a_j^\dagger a_j$$

acting on boson Fock space over $\mathbb{C}^{n+1}$ where each $w_j > 0$ $(0 \leq j \leq n)$ and a_j and $a_j^\dagger$ are the annihilation and creation operators (respectively) associated to the jth oscillator. These satisfy the discrete form of the CCR relations

$$[a_j, a_k] = 0 \qquad [a_j, a_k^\dagger] = \delta_{jk} I \qquad 0 \leq j, k \leq n. \tag{1}$$

The system is initially in the equilibrium state described by the Gibbs density operator

$$\rho = \frac{1}{Z_{n+1}} \mathrm{e}^{-\beta H} \tag{2}$$

where β is an inverse temperature parameter and $Z_{n+1} = \mathrm{Tr}(\mathrm{e}^{-\beta H})$†.

In general, we will use the notation $\Gamma(h)$ to denote boson Fock space over the complex Hilbert space h. Making the natural identification

$$\Gamma(\mathbb{C}^{n+1}) = \Gamma(\mathbb{C}) \otimes \Gamma(\mathbb{C}^n)$$

we have

$$H = H_0 + H_E \qquad \rho = \rho_0 \otimes \rho_E$$

where

$$H_0 = w_0 a_0^\dagger a_0 \otimes I \qquad \rho_0 = (\mathrm{e}^{-\beta H_0}/Z_0) \otimes I$$

$$H_E = I \otimes \sum_{j=1}^{n} a_j^\dagger a_j \qquad \rho_E = \frac{1}{Z_n} I \otimes \mathrm{e}^{-\beta H_E}.$$

We may now regard our set-up as describing, for example, the siting of one mode of a laser field within the reservoir provided by its pumping mechanism (see, e.g., [1, 2]). To describe the field-reservoir interaction, we introduce the Hamiltonian

$$H_I = \sum_{j=1}^{n} [g_j(\varepsilon a_0 + \eta a_0^\dagger) a_j + \bar{g}_j(\varepsilon a_0^\dagger + \eta a_0) a_j^\dagger] \tag{3}$$

where $g_j \in \mathbb{C}$ $(1 \leq j \leq n)$ are coupling parameters, ε and η are non-negative constants and we have suppressed the tensor product signs for notational convenience. We note that (3) is the most general bilinear interaction between the systems. The interaction Hamiltonian for the laser-reservoir model is as in (3) with $\varepsilon = \eta = 1$. Subsequently the analysis of [1] introduces the 'rotating wave approximation', which amounts to putting $\varepsilon = 0$ in (3), thus eliminating those terms in H_I which fail to commute with the total number operator. We will not implement this procedure here and the role of the constants η and ε is to keep track, in the following, of the 'rotating wave' and 'anti-rotating wave' terms, respectively.

† We work in a system of units wherein Planck's constant has value 2π and Boltzmann's constant is 1.

3. Approximation of the interaction dynamics by a stochastic evolution equation

Working in the interaction picture, we aim to describe the evolution of our system by means of a family of unitary operators $\{W(t), t \geqslant 0\}$ on $\Gamma(\mathbb{C})\otimes\Gamma(\mathbb{C}^n)$ satisfying the differential equation

$$\begin{aligned} &\frac{\mathrm{d}W(t)}{\mathrm{d}t} = -\mathrm{i}\,W(t)\{\exp[\mathrm{i}t(H_0+H_E)]H_I\,\exp[-\mathrm{i}t(H_0+H_E)]\} \\ &W(0)=I \end{aligned} \tag{4}$$

i.e.

$$\begin{aligned} &\mathrm{d}W(t)/\mathrm{d}t = -\mathrm{i}\,W(t)K(t) \\ &W(0)=I \end{aligned} \tag{5}$$

where

$$K(t) = G(t)^\dagger F(t) + G(t)F(t)^\dagger$$

with

$$G(t) = \eta a\,\exp(\mathrm{i}w_0 t) + \varepsilon a^\dagger \exp(-\mathrm{i}w_0 t)$$

and

$$F(t) = \sum_{j=1}^{n} g_j \exp(-\mathrm{i}w_0 t)a_j \qquad t \geqslant 0.$$

A standard technique for obtaining approximate solutions to (5) is to introduce the Wigner-Weisskopf approximation wherein we assume that all the g_j are close to a common value (which we take to be 1) and the w_j are distributed over a wide frequency range. The effect of the approximation is to replace the atomic reservoir with a system which has infinitely many degrees of freedom so that $\{F(t), F^\dagger(t), t \geqslant 0\}$ behaves like 'quantum white noise'. The analysis is identical to that of [3] for a two-level atom in a radiation field (see also [4, 5]).

Specifically, (5) is approximated by the quantum stochastic differential equation (SDE) [6]

$$\begin{aligned} \mathrm{d}\hat{W}(t) &= \hat{W}(t)(-\mathrm{i}G^\dagger(t)\,\mathrm{d}A - \mathrm{i}G(t)\,\mathrm{d}A^\dagger) - \tfrac{1}{2}(\lambda^2 G^\dagger(t)G(t) + \tfrac{1}{2}\mu^2 G(t)G(t)^\dagger)\,\mathrm{d}t \\ &\hat{W}(0) = I \end{aligned} \tag{6}$$

where

$$\lambda^2 = \frac{1}{1-\exp(-\beta w_0)} \qquad \mu^2 = \frac{\exp(-\beta w_0)}{1-\exp(-\beta w_0)}$$

$\mathrm{d}A$ and $\mathrm{d}A^\dagger$ are stochastic differentials of quantum Brownian motion of variance $\sigma^2 = \lambda^2 + \mu^2$ [7] and $\{\hat{W}(t), t \geqslant 0\}$ are a family of unitary operators on $\boldsymbol{H} = \Gamma(\mathbb{C})\otimes\boldsymbol{H}^{\text{noise}}$ where

$$\boldsymbol{H}^{\text{noise}} = \Gamma(L^2(\mathbb{R}))\otimes\Gamma(\overline{L^2(\mathbb{R})}).$$

(If h is a complex Hilbert space, we use the notation $\bar{h}$ to denote its dual.)

As in [3], we enlarge the dynamics as given by (6) to include the free evolution on $\Gamma(\mathbb{C})$ by writing $U(t) = \hat{W}(t)V(t)$ where $V(t) = \exp(\mathrm{i}tH_0)$. For convenience, we also transform the quantum noise to the equivalent form $dA \to i\,dA$, $dA^\dagger \to -i\,dA^\dagger$. We then obtain

$$\begin{aligned} dU(t) &= U(t)[G^\dagger\,dA - G\,dA^\dagger + (iH_0 - \tfrac{1}{2}\lambda^2 G^\dagger G - \tfrac{1}{2}\mu^2 GG^\dagger)\,dt] \\ U(0) &= I \end{aligned} \tag{7}$$

where $G = G(0)$ (cf [6, 7]).

4. Ornstein-Uhlenbeck type behaviour of the evolution

Let $a(t) = U(t)a_0U(t)^\dagger (t \geqslant 0)$. We obtain, by stochastic differentiation, the SDE

$$da = \varepsilon\,dA^\dagger - \eta\,dA - [iw_0 + \tfrac{1}{2}(\eta^2 - \varepsilon^2)]a\,dt \tag{8}$$

whose solution is given by

$$a(t) = g_t(0)a_0 + \varepsilon \int_0^t g_t(\tau)\,dA^\dagger - \eta \int_0^t g_t(\tau)\,dA \tag{9}$$

where, for $t \geqslant 0$, $0 \leqslant \tau \leqslant t$,

$$g_t(\tau) = \exp\{-[iw_0 + \tfrac{1}{2}(\eta^2 - \varepsilon^2)](t - \tau)\}. \tag{10}$$

If we make the rotating wave approximation ($\varepsilon = 0$), (9) is the quantum Ornstein-Uhlenbeck process of [6, 8, 17]. The anti-Ornstein-Uhlenbeck process of [6] is obtained by taking $\eta = 0$ in (9).

Noting that $dX = -i(dA - dA^\dagger)$ is a realisation of classical Brownian motion, we obtain in (9) for the case $\varepsilon = \eta = 1$,

$$a(t) = \exp(-iw_0t)a_0 - i \int_0^t \exp[-iw_0(t - \tau)]\,dX \tag{11}$$

which is reminiscent of a classical Ornstein-Uhlenbeck process [9] in imaginary time.

For $f \in L^2(\mathbb{R})$, we define operators in $\boldsymbol{H}^{\text{noise}}$

$$\begin{aligned} A(f) &= \lambda a(f) \otimes I + \mu I \otimes \bar{a}^\dagger(\bar{f}) \\ A^\dagger(f) &= \lambda a^\dagger(f) \otimes I + \mu I \otimes \bar{a}(\bar{f}) \end{aligned} \tag{12}$$

where $a(f)$ and $a^\dagger(f)$ are annihilation and creation operators, respectively, in $\Gamma(L^2(\mathbb{R}))$. (In general, if T is an operator on h, $\bar{T}$ is that operator on $\bar{h}$ for which $\bar{T}\bar{f} = \overline{Tf}$, for f in the domain of T.) The operators (12) satisfy the boson commutation relations [10],

$$[A(f), A(g)] = 0 \qquad [A(f), A^\dagger(g)] = \langle f, g\rangle I \tag{13}$$

for each $f, g \in L^2(\mathbb{R})$, with the maps $f \to A(f)$, $f \to A^\dagger(f)$ being conjugate-linear and linear (respectively).

We note [7] that dA and $dA^\dagger$ are the differentials of $A(\chi_{[0,t)})$ and $A^\dagger(\chi_{[0,t)})$ respectively (where χ_A is the indicator function of the subset $A \subseteq \mathbb{R}$) and we will use below the fact that we can write (9) in the form

$$a(t) = g_t(0)a_0 + A^\dagger(\varepsilon g_t \chi_{[0,t)}) - A(\eta \bar{g}_t \chi_{[0,t)}). \tag{14}$$

We will also need some facts about Weyl operators. These are a family of unitary operators $\{W(f), f \in h\}$, where h is a complex Hilbert space, acting on a complex Hilbert space $\boldsymbol{h}$ and satisfying the relations [10]

$$\begin{aligned} W(f)^* &= W(-f) \\ W(f+g) &= W(f)\,W(g)\exp(-\mathrm{i}\,\mathrm{Im}\langle f, g\rangle) \end{aligned} \tag{15}$$

for all $f, g \in h$. Two examples of importance for us are

(i) $h = \mathbb{C}$, $\boldsymbol{h} = \Gamma(\mathbb{C})$

$$W(z) = \exp(\bar{z}a_0 - za_0^\dagger) \qquad z \in \mathbb{C} \tag{16}$$

(ii) $h = L^2(\mathbb{R})$, $\boldsymbol{h} = \boldsymbol{H}^{\text{noise}}$

$$W(f) = \exp(A(f) - A^\dagger(f)) \qquad f \in L^2(\mathbb{R}). \tag{17}$$

5. Description of the reduced dynamics

Let $\boldsymbol{N}$ be the von Neumann subalgebra of $B(\boldsymbol{H}^{\text{noise}})$ generated by $\{W(f), f \in L^2(\mathbb{R})\}$. We define the vacuum conditional expectation $\boldsymbol{E}_0 : B(\Gamma(\mathbb{C})) \otimes \boldsymbol{N} \to B(\Gamma(\mathbb{C}))$ by continuous linear extension of

$$\boldsymbol{E}_0(X \otimes W(f)) = \exp(-\tfrac{1}{2}\sigma^2\|f\|^2)X \tag{18}$$

for $X \in B(\Gamma(\mathbb{C}))$ [7].

The prescription

$$T_t(X) = \boldsymbol{E}_0(U(t)XU(t)^\dagger) \tag{19}$$

yields a completely positive, identity preserving one-parameter semigroup $\{T_t, t \geq 0\}$ of operators in $B(h_0)$ [6, 7] whose infinitesimal generator is given by

$$L(X) = \mathrm{i}[H_0, X] + \lambda^2[G^\dagger XG - \tfrac{1}{2}\{G^\dagger G, X\}] + \mu^2[GXG^\dagger - \tfrac{1}{2}\{GG^\dagger, X\}] \tag{20}$$

for $X \in B(\Gamma(\mathbb{C}))$ (cf [11]).

We regard this semigroup as a description of the diffusion of the oscillator through the reservoir.

Let ω_0 be the state on $B(\Gamma(\mathbb{C}))$ given by

$$\omega_0(X) = \mathrm{Tr}\,\rho_0 X \qquad X \in B(\Gamma(\mathbb{C})). \tag{21}$$

We investigate the conditions under which ω_0 is left invariant by the semigroup action, i.e.

$$\omega_0(T_t(X)) = \omega_0(X) \qquad \forall t \geq 0$$

$X \in B(\Gamma(\mathbb{C}))$. A necessary and sufficient condition [15] for this is that the semigroup satisfies the quantum detailed balance condition of [12] with respect to ω_0, i.e. there exists another semigroup $\{T_t^\dagger, t \geq 0\}$ on $B(\Gamma(\mathbb{C}))$ such that

$$\omega_0(T_t^\dagger(X)Y) = \omega_0(XT_t(Y))$$

for all $t \geq 0$, $X, Y \in B(\Gamma(\mathbb{C}))$ and

$$L(X) - L^\dagger(X) = 2\mathrm{i}[H_0, X]$$

for all $X \in B(\Gamma(\mathbb{C}))$ where $L^\dagger$ is the infinitesimal generator of $\{T_t^\dagger, t \geq 0\}$.

In our case, the particular form of L (given by (20)) ensures that the quantum detailed balance condition is equivalent to the requirement [13] that

$$V(t)GV(t)^* = \exp(\mathrm{i}t\beta\omega_0)G \Leftrightarrow \varepsilon = 0$$

i.e. the rotating wave approximation is made.

Since ω_0 is not left invariant by the semigroup in general, we compute its time evolution $\{\omega_t, t \geqslant 0\}$ where for $t \geqslant 0$, $X \in B(\Gamma(\mathbb{C}))$ we define

$$\begin{aligned} \omega_t(X) &= \operatorname{Tr} \rho_t X \\ &= \operatorname{Tr}(T_t^* \rho) X \\ &= \operatorname{Tr} \rho T_t(X) \end{aligned} \tag{22}$$

where $\{T_t^*, t \geqslant 0\}$ is a positive trace-preserving semigroup on the Banach space of trace class operators on $\Gamma(\mathbb{C})$.

Since $\{W(z), z \in \mathbb{C}\}$ acts irreducibly on $\Gamma(\mathbb{C})$, it is sufficient to compute (22) on this set. We note that

$$\omega_0(W(z)) = \exp(-\tfrac{1}{2}\sigma^2|z|^2) \qquad z \in \mathbb{C}. \tag{23}$$

By (19) and (16) we obtain

$$\begin{aligned} T_t(W(z)) &= \boldsymbol{E}_0(U(t)\exp(\bar{z}a_0 - za_0^\dagger)U(t)^\dagger) \\ &= \boldsymbol{E}_0\{\exp[\bar{z}(g_t(0)a_0 + A^\dagger(\varepsilon g_t\chi_{[0,t)}) - A(\eta\bar{g}_t\chi_{[0,t)})) \\ &\quad - z(\overline{g_t(0)}a_0^\dagger + A(\varepsilon g_t\chi_{[0,t)}) - A^\dagger(\eta\bar{g}_t\chi_{[0,t)}))]\} && \text{by (14)} \\ &= \boldsymbol{E}_0(W(g_t(0)z)W[-(\eta z\bar{g}_t + \varepsilon\bar{z}g_t)\chi_{[0,t)}]) && \text{by (16) and (17)} \\ &= W(g_t(0)z)\exp[-\tfrac{1}{2}\sigma^2\|(\eta z\bar{g}_t + \varepsilon\bar{z}g_t)\chi_{[0,t)}\|^2] && \text{by (18).} \end{aligned}$$

Thus by (23) we obtain

$$\omega_t(W(z)) = \exp(-\tfrac{1}{2}\sigma(t)^2|z|^2) \tag{24}$$

where

$$\sigma(t)^2 = \sigma^2(\|(\eta\,\mathrm{e}^{\mathrm{i}\theta}\bar{g}_t + \varepsilon\,\mathrm{e}^{-\mathrm{i}\theta}g_t)\chi_{[0,t)}\|^2 + |g_t(0)|^2) \tag{25}$$

and $\theta = \arg z$. There are two cases to consider.

(i) For $\varepsilon \neq \eta$, $t \geqslant 0$

$$\begin{aligned} \sigma(t)^2 = \sigma^2\bigg(&\exp[-(\eta^2-\varepsilon^2)t] + \frac{\eta^2+\varepsilon^2}{\eta^2-\varepsilon^2}\{1-\exp[-(\eta^2-\varepsilon^2)t]\} \\ &+ \frac{2\eta\epsilon}{(\eta^2-\varepsilon^2)^2+4w_0^2}[\![(\eta^2-\varepsilon^2)\{\cos 2\theta - \exp[-(\eta^2-\varepsilon^2)t]\cos 2(\theta+\omega_0 t)\} \\ &- 2w_0\{\sin 2\theta - \exp[-(\eta^2-\varepsilon^2)t]\sin 2(\theta+w_0 t)\}]\!]\bigg). \end{aligned} \tag{26}$$

Each state ω_t $(t > 0)$, although quasi-free, is no longer gauge invariant [10]. Note that if we put $\varepsilon = 0$ in (26) we obtain $\sigma(t) = \sigma$, for all $t \geqslant 0$ as we expect from the validity of quantum detailed balance in this case. In general, the sequence of states $(\omega_t, t \geq 0)$ converges (in the weak * topology) to ω_∞ where

$$\omega_\infty(W(z)) = \exp(-\tfrac{1}{2}\sigma_\infty|z|^2)$$

where

$$\sigma_\infty^2 = \sigma^2\left(\frac{\eta^2+\varepsilon^2}{\eta^2-\varepsilon^2} + \frac{2\eta\varepsilon}{(\eta^2-\varepsilon^2)^2+4w_0^2}[(\eta^2-\varepsilon^2)\cos 2\theta - 2w_0 \sin 2\theta]\right). \quad (27)$$

(ii) For $\varepsilon = \eta$, $t \geq 0$

$$\sigma(t)^2 = \sigma^2\left(1 + 2\eta^2 t + \frac{\eta^2}{w_0}[\sin 2(\theta + w_0 t) - \sin 2\theta]\right). \quad (28)$$

Again, the quasi-free states $(\omega_t, t > 0)$ are not gauge invariant, convergence in this case being to the (infinite-temperature) central state

$$\begin{aligned} &\omega_\infty(W(z)) = 0 \qquad z \neq 0 \\ &\omega_\infty(I) = 1. \end{aligned} \quad (29)$$

The expressions $\sigma(t)^2$ in (26) and (28) and σ_∞^2 in (27) cannot be read as variances of the characteristic functionals $\omega_t(W(z))$ and $\omega_\infty(W(z))$ owing to their specific dependence on z through θ.

The passage from equations (5) to (6) is essentially a singular coupling limit [13]. In this regard it is interesting to compare (26) and (28) with the results obtained in [14] where the weak coupling limit is taken.

We note that the quantum stochastic process arising from (7) satisfies the Markov property [15] in contrast to the model considered in [16].

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References

[1] Sargent M, Scully M O and Lamb W E 1974 *Laser Physics* (New York: Addison-Wesley)
[2] Hepp K and Lieb E H 1975 *The Laser: A Reversible Quantum Dynamical System with Irreversible Classical Macroscopic Motion* (*Lecture Notes in Maths* **38**) (Berlin: Springer)
[3] Applebaum D 1986 *J. Math. Phys.* to be published
[4] Accardi L 1985 *Statistical Physics and Dynamical Systems, Rigorous Results*; *2nd Colloq.* (Basel: Birkhauser)
[5] von Waldenfels W 1985 *Stochastic Aspects of Classical and Quantum Systems* (*Lecture Notes in Maths* **1109**) (Berlin: Springer) p 155
[6] Hudson R L and Parthasarathy K R 1982 *Quantum Probability and Applications to the Quantum Theory of Irreversible Processes* (*Lecture Notes in Maths* **1055**) (Berlin: Springer) p 173; 1984 *Commun. Math. Phys.* **93** 301
[7] Hudson R L and Lindsay J M 1985 *Ann. Inst. H Poincaré* **43** 133
[8] Streater R F 1982 *J. Phys. A: Math. Gen.* **15** 1477
[9] Ikeda N and Watanabe S 1981 *Stochastic Differential Equations and Diffusion Processes* (Amsterdam: North-Holland/Kodansha)

[10] Bratteli O and Robinson D W 1979 *Operator Algebras and Quantum Statistical Mechanics* vol II (Berlin: Springer)
[11] Lindblad G 1976 *Commun. Math. Phys.* **48** 119
[12] Kossakowski A, Frigerio A, Gorini V and Verri M 1977 *Commun. Math. Phys.* **57** 97
[13] Frigerio A and Gorini V 1984 *Commun. Math. Phys.* **93** 517
[14] Davies E B 1973 *Commun. Math. Phys.* **33** 171
[15] Frigerio A 1985 *Construction of Stationary Quantum Processes Through Quantum Stochastic Calculus in Quantum Probability and Applications II* (*Lecture Notes in Maths* **1136**) (Berlin: Springer) p 207
[16] Ford G W, Kac M and Mazur P 1965 *J. Math. Phys.* **6** 504
[17] Hudson R L, Ion P D F and Parthasarathy K R 1982 *Commun. Math. Phys.* **83** 261